

Existence of nontrivial solutions for periodic Schrödinger equations with new nonlinearities

Shaowei Chen ^{*} Dawei Zhang

*School of Mathematical Sciences, Huaqiao University,
Quanzhou 362021, China*

Abstract: We study the Schrödinger equation:

$$-\Delta u + V(x)u + f(x, u) = 0, \quad u \in H^1(\mathbb{R}^N),$$

where V is periodic and f is periodic in the x -variables, 0 is in a gap of the spectrum of the operator $-\Delta + V$. We prove that under some new assumptions for f , this equation has a nontrivial solution. Our assumptions for the nonlinearity f are very weak and greatly different from the known assumptions in the literature.

Key words: Semilinear Schrödinger equations; periodic potentials; generalized linking theorem.

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1 Introduction and statement of results

In this paper, we consider the following Schrödinger equation:

$$-\Delta u + V(x)u + f(x, u) = 0, \quad u \in H^1(\mathbb{R}^N), \quad (1.1)$$

where $N \geq 1$. For V and f , we assume

- (v). $V \in C(\mathbb{R}^N)$ is 1-periodic in x_j for $j = 1, \dots, N$, 0 is in a spectral gap $(-\mu_{-1}, \mu_1)$ of $-\Delta + V$ and $-\mu_{-1}$ and μ_1 lie in the essential spectrum of $-\Delta + V$.

Denote

$$\mu_0 := \min\{\mu_{-1}, \mu_1\}.$$

- (f₁). $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in x_j for $j = 1, \dots, N$. And there exist constants $C > 0$ and $2 < p < 2^*$ such that

$$|f(x, t)| \leq C(1 + |t|^{p-1}), \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$$

$$\text{where } 2^* := \begin{cases} \frac{2N}{N-2}, & N \geq 3 \\ \infty, & N = 1, 2. \end{cases}$$

- (f₂). The limit $\lim_{t \rightarrow 0} f(x, t)/t = 0$ holds uniformly for $x \in \mathbb{R}^N$. And there there exists $D > 0$ such that

$$\inf_{x \in \mathbb{R}^N, |t| \geq D} \frac{f(x, t)}{t} > \max_{\mathbb{R}^N} V_-. \quad (1.2)$$

where $V_{\pm}(x) = \max\{\pm V(x), 0\}$, $\forall x \in \mathbb{R}^N$.

^{*}E-mail address: swchen6@163.com (Shaowei Chen)

(f₃). For any $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, $\tilde{F}(x, t) \geq 0$, where

$$\tilde{F}(x, t) := \frac{1}{2}tf(x, t) - F(x, t)$$

and $F(x, t) = \int_0^t f(x, s)ds$.

(f₄). There exist $0 < \kappa < D$ and $\nu \in (0, \mu_0)$ such that, for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| < \kappa$,

$$|f(x, t)| \leq \nu|t| \quad (1.3)$$

and for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ with $\kappa \leq |t| \leq D$,

$$\tilde{F}(x, t) > 0. \quad (1.4)$$

Remark 1.1. By the definitions of F and \tilde{F} , it is easy to verify that, for all $(x, t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$,

$$\frac{\partial}{\partial t} \left(\frac{F(x, t)}{t^2} \right) = \frac{2\tilde{F}(x, t)}{t^3}.$$

Together with $f(x, t) = o(t)$ as $|t| \rightarrow 0$ and (f₃), this implies that

$$F(x, t) \geq 0 \text{ for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}. \quad (1.5)$$

A solution u of (1.1) is called nontrivial if $u \not\equiv 0$. Our main results are as follows:

Theorem 1.2. Suppose (v), and (f₁) – (f₄) are satisfied. Then Eq.(1.1) has a nontrivial solution.

Note that

(f'₂). The limits $\lim_{t \rightarrow 0} f(x, t)/t = 0$ and $\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{t} = +\infty$ hold uniformly for $x \in \mathbb{R}^N$.

implies (f₂). We have the following corollary

Corollary 1.3. Suppose (v), (f₁), (f'₂), (f₃), and (f₄) are satisfied. Then Eq.(1.1) has a nontrivial solution.

It is easy to verify that the condition

(f'₄). $\tilde{F}(x, t) > 0$ for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

and the assumption that $f(x, t)/t \rightarrow 0$ as $t \rightarrow 0$ uniformly for $x \in \mathbb{R}^N$ imply (f₃) and (f₄). Therefore, we have the following corollary:

Corollary 1.4. Suppose (v), (f₁), (f₂), and (f'₄) are satisfied. Then Eq.(1.1) has a nontrivial solution.

Semilinear Schrödinger equations with periodic coefficients have attracted much attention in recent years due to its numerous applications. One can see [1]-[6], [8], [10]-[17], [20]-[28] and the references therein. In [2], the authors used the dual variational method to obtain a nontrivial solution of (1.1) with $f(x, t) = \pm W(x)|t|^{p-2}t$, where W is a asymptotically periodic function. In [24], Troestler and Willem firstly obtained nontrivial solutions for (1.1) with f is a C^1 function satisfying the Ambrosetti-Rabinowitz condition:

$$(AR) \quad \text{there exists } \alpha > 2 \text{ such that for every } u \neq 0, 0 < \alpha G(x, u) \leq g(x, u)u,$$

where $g(x, u) = -f(x, u)$ and $G(x, u) = -F(x, u)$, and

$$\left| \frac{\partial f(x, u)}{\partial u} \right| \leq C(|u|^{p-2} + |u|^{q-2})$$

with $2 < p < q < 2^*$. Then, in [11], Kryszewski and Szulkin developed some infinite-dimensional linking theorems. Using these theorems, they improved Troestler and Willem's results and obtained nontrivial solutions for (1.1) with f only satisfying (f_1) and the (AR) condition. These generalized linking theorems were also used by Li and Szulkin to obtain nontrivial solution for (1.1) under some asymptotically linear assumptions for f (see [13]). In [16] (see also [17]), existence of nontrivial solutions for (1.1) under (f_1) and the (AR) condition was also obtained by Pankov and Pflüger through approximating (1.1) by a sequence of equations defined in bounded domains. In the celebrated paper [21], Schechter and Zou combined a generalized linking theorem with the monotonicity methods of Jeanjean (see [10]). They obtained a nontrivial solution of (1.1) when f exhibits the critical growth. A similar approach was applied by Szulkin and Zou to obtain homoclinic orbits of asymptotically linear Hamiltonian systems (see [23]). Moreover, in [5] (see also [6]), Li and Ding obtained nontrivial solutions for (1.1) under some new superlinear assumptions on f different from the classical (AR) conditions.

Our assumptions on f are very weak and greatly different from the assumptions mentioned above. In fact, our assumptions $(f_1) - (f_4)$ do not involve the properties of f at infinity. It may be asymptotically linear growth at infinity, i.e., $\limsup_{|t| \rightarrow \infty} \frac{f(x,t)}{t} < +\infty$ or superlinear growth at infinity as well, i.e., $\liminf_{|t| \rightarrow \infty} \frac{f(x,t)}{t} = +\infty$.

In this paper, we use the generalized linking theorem for a class of parameter-dependent functionals (see [21, Theorem 2.1] or Proposition 2.3 in the present paper) to obtain a sequence of approximate solutions for (1.1). Then, we prove that these approximate solutions are bounded in $L^\infty(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ (see Lemma 3.1 and 3.2). Finally, using the concentration-compactness principle, we obtain a nontrivial solution of (1.1).

Notation. $B_r(a)$ denotes the open ball of radius r and center a . For a Banach space E , we denote the dual space of E by E' , and denote strong and weak convergence in E by \rightarrow and \rightharpoonup , respectively. For $\varphi \in C^1(E; \mathbb{R})$, we denote the Fréchet derivative of φ at u by $\varphi'(u)$. The Gateaux derivative of φ is denoted by $\langle \varphi'(u), v \rangle$, $\forall u, v \in E$. $L^p(\mathbb{R}^N)$ denotes the standard L^p space ($1 \leq p \leq \infty$), and $H^1(\mathbb{R}^N)$ denotes the standard Sobolev space with norm $\|u\|_{H^1} = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx)^{1/2}$. We use $O(h)$, $o(h)$ to mean $|O(h)| \leq C|h|$, $o(h)/|h| \rightarrow 0$ as $|h| \rightarrow 0$.

2 Existence of approximate solutions for Eq.(1.1)

Under the assumptions (v) , (f_1) , and (f_2) , the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx + \int_{\mathbb{R}^N} F(x, u) dx \quad (2.1)$$

is of class C^1 on $X := H^1(\mathbb{R}^N)$, and the critical points of Φ are weak solutions of (1.1).

Assume that (v) holds, and let $S = -\Delta + V$ be the self-adjoint operator acting on $L^2(\mathbb{R}^N)$ with domain $D(S) = H^2(\mathbb{R}^N)$. By virtue of (v) , we have the orthogonal decomposition

$$L^2 = L^2(\mathbb{R}^N) = L^+ + L^-$$

such that S is negative (resp. positive) in L^- (resp. in L^+). As in [5, Section 2] (see also [6, Chapter 6.2]), let $X = D(|S|^{1/2})$ be equipped with the inner product

$$(u, v) = (|S|^{1/2}u, |S|^{1/2}v)_{L^2}$$

and norm $\|u\| = \||S|^{1/2}u\|_{L^2}$, where $(\cdot, \cdot)_{L^2}$ denotes the inner product of L^2 . From (v) ,

$$X = H^1(\mathbb{R}^N)$$

with equivalent norms. Therefore, X continuously embeds in $L^q(\mathbb{R}^N)$ for all $2 \leq q \leq 2N/(N-2)$ if $N \geq 3$ and for all $q \geq 2$ if $N = 1, 2$. In addition, we have the decomposition

$$X = X^+ + X^-,$$

where $X^\pm = X \cap L^\pm$ is orthogonal with respect to both $(\cdot, \cdot)_{L^2}$ and (\cdot, \cdot) . Therefore, for every $u \in X$, there is a unique decomposition

$$u = u^+ + u^-, \quad u^\pm \in X^\pm$$

with $(u^+, u^-) = 0$ and

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x) u^2 dx = \|u^+\|^2 - \|u^-\|^2, \quad u \in X. \quad (2.2)$$

Moreover,

$$\mu_{-1} \|u^-\|_{L^2}^2 \leq \|u^-\|^2, \quad \forall u \in X, \quad (2.3)$$

and

$$\mu_1 \|u^+\|_{L^2}^2 \leq \|u^+\|^2, \quad \forall u \in X. \quad (2.4)$$

The functional Φ defined by (2.1) can be rewritten as

$$\Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) + \psi(u), \quad (2.5)$$

where

$$\psi(u) = \int_{\mathbb{R}^N} F(x, u) dx.$$

The above variational setting for the functional (2.1) is standard. One can consult [5] or [6] for more details.

Let $\{e_k^\pm\}$ be the total orthonormal sequence in X^\pm . Let $P : X \rightarrow X^-$, $Q : X \rightarrow X^+$ be the orthogonal projections. We define

$$|||u||| = \max \left\{ \|Pu\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} |(Qu, e_k^+)| \right\}$$

on X . The topology generated by $||| \cdot |||$ is denoted by τ , and all topological notation related to it will include this symbol.

Lemma 2.1. *Suppose that (v) holds. Then*

- (a). $\max_{\mathbb{R}^N} V_- \geq \mu_{-1}$, where μ_{-1} is defined in (v).
- (b). For any $C > \mu_{-1}$, there exists $u_0 \in X^-$ with $\|u_0\| = 1$ such that $C \|u_0\|_{L^2} > 1$.

Proof. (a). We apply an indirect argument, and assume by contradiction that

$$c := \max_{\mathbb{R}^N} V_- < \mu_{-1}.$$

From assumption (v), $-\mu_{-1}$ is in the essential spectrum of the operator (with domain $D(L) = H^2(\mathbb{R}^N)$)

$$L = -\Delta + V : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$$

Then, by the Weyl's criterion (see, for example, [18, Theorem VII.12] or [9, Theorem 7.2]), there exists a sequence $\{u_n\} \subset H^2(\mathbb{R}^N)$ with the properties that $\|u_n\|_{L^2} = 1$, $\forall n$ and $\|-\Delta u_n + V u_n + \mu_{-1} u_n\|_{L^2} \rightarrow 0$.

Since $\mu_{-1} > c = \max_{\mathbb{R}^N} V_-$, we deduce that $-V_-(x) + \mu_{-1} > 0$ for all $x \in \mathbb{R}^N$. Together with the facts that V is a continuous periodic function and $V = V_+ - V_-$, this implies

$$\inf_{x \in \mathbb{R}^N} (V(x) + \mu_{-1}) > 0.$$

It follows that there exists a constant $C' > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + \mu_{-1})u^2)dx \geq C'\|u\|^2, \quad \forall u \in X. \quad (2.6)$$

Note that

$$\int_{\mathbb{R}^N} (-\Delta u_n + V(x)u_n + \mu_{-1}u_n)u_n dx = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + (V(x) + \mu_{-1})u_n^2)dx.$$

Together with (2.6) and the fact that $\|-\Delta u_n + V(x)u_n + \mu_{-1}u_n\|_{L^2} \rightarrow 0$ and $\|u_n\|_{L^2} = 1$, this implies $\|u_n\| \rightarrow 0$. It contradicts $\|u_n\|_{L^2} = 1, \forall n$. Therefore, $\max_{\mathbb{R}^N} V_- \geq \mu_{-1}$.

(b). It suffices to prove that

$$\mu_{-1} = C_- := \inf\{\|u\|^2 \mid u \in X^-, \|u\|_{L^2} = 1\}.$$

From (2.3), we deduce that $\mu_{-1} \leq C_-$. From assumption (v), $-\mu_{-1}$ is in the essential spectrum of L . By the Weyl's criterion, there exists $\{u_n\} \subset H^2(\mathbb{R}^N)$ such that $\|u_n\|_{L^2} = 1$ and $\|-\Delta u_n + V(x)u_n + \mu_{-1}u_n\|_{L^2} \rightarrow 0$. Multiplying $-\Delta u_n + V(x)u_n + \mu_{-1}u_n$ by u_n^+ and then integrating on \mathbb{R}^N , by (2.2) and (2.4), we get that

$$\begin{aligned} (\mu_1 + \mu_{-1})\|u_n^+\|_{L^2}^2 &\leq \int_{\mathbb{R}^N} (|\nabla u_n^+|^2 + V(x)(u_n^+)^2 + \mu_{-1}(u_n^+)^2)dx \\ &= \int_{\mathbb{R}^N} (-\Delta u_n + V(x)u_n + \mu_{-1}u_n)u_n^+ dx \rightarrow 0. \end{aligned}$$

It follows that $\|u_n^-\|_{L^2} \rightarrow 1$. Multiplying $-\Delta u_n + V(x)u_n + \mu_{-1}u_n$ by u_n^- and then integrating on \mathbb{R}^N , we get that

$$\begin{aligned} -\|u_n^-\|_{L^2}^2 + \mu_{-1}\|u_n^-\|_{L^2}^2 &= \int_{\mathbb{R}^N} (|\nabla u_n^-|^2 + V(x)(u_n^-)^2 + \mu_{-1}(u_n^-)^2)dx \\ &= \int_{\mathbb{R}^N} (-\Delta u_n + V(x)u_n + \mu_{-1}u_n)u_n^- dx \rightarrow 0. \end{aligned} \quad (2.7)$$

It implies that $\mu_{-1} \geq C_-$. This together with $\mu_{-1} \leq C_-$ implies $\mu_{-1} = C_-$. \square

Let $R > r > 0$ and

$$A := \inf_{x \in \mathbb{R}^N, |t| \geq D} \frac{f(x, t)}{t}.$$

From assumption (1.2), we have $A > \max_{\mathbb{R}^N} V_-$. Together with the result (a) of Lemma 2.1, this implies that $\frac{1}{2}(A + \mu_{-1}) > \mu_{-1}$. Choose

$$\gamma \in (\mu_{-1}, (A + \mu_{-1})/2). \quad (2.8)$$

Then by the result (b) of Lemma 2.1, there exists $u_0 \in X^-$ with $\|u_0\| = 1$ such that

$$\gamma\|u_0\|_{L^2} > 1. \quad (2.9)$$

Set

$$N = \{u \in X^- \mid \|u\| = r\}, \quad M = \{u \in X^+ \oplus \mathbb{R}^+ u_0 \mid \|u\| \leq R\}.$$

Then, M is a submanifold of $X^+ \oplus \mathbb{R}^+ u_0$ with boundary

$$\partial M = \{u \in X^- \mid \|u\| \leq R\} \cup \{u^- + tu_0 \mid u^- \in X^-, t > 0, \|u^- + tu_0\| = R\}.$$

Definition 2.2. Let $\phi \in C^1(X; \mathbb{R})$. A sequence $\{u_n\} \subset X$ is called a Palais-Smale sequence at level c ($(PS)_c$ -sequence for short) for ϕ , if $\phi(u_n) \rightarrow c$ and $\|\phi'(u_n)\|_{X'} \rightarrow 0$ as $n \rightarrow \infty$.

The following proposition is proved in [21] (see [21, Theorem 2.1]).

Proposition 2.3. *Let $0 < K < 1$. The family of C^1 -functional $\{H_\lambda\}$ has the form*

$$H_\lambda(u) = \lambda I(u) - J(u), \quad u \in X, \quad \lambda \in [K, 1]. \quad (2.10)$$

Assume

- (a) $J(u) \geq 0, \forall u \in X$,
- (b) $|I(u)| + J(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$,
- (c) for all $\lambda \in [K, 1]$, H_λ is τ -sequentially upper semi-continuous, i.e., if $\|u_n - u\| \rightarrow 0$, then

$$\limsup_{n \rightarrow \infty} H_\lambda(u_n) \leq H_\lambda(u),$$

and H'_λ is weakly sequentially continuous. Moreover, H_λ maps bounded sets to bounded sets,

- (d) there exist $u_0 \in X^- \setminus \{0\}$ with $\|u_0\| = 1$, and $R > r > 0$ such that for all $\lambda \in [K, 1]$,

$$\inf_N H_\lambda > \sup_{\partial M} H_\lambda.$$

Then there exists $E \subset [K, 1]$ such that the Lebesgue measure of $[K, 1] \setminus E$ is zero and for every $\lambda \in E$, there exist c_λ and a bounded $(PS)_{c_\lambda}$ -sequence for H_λ , where c_λ satisfies

$$\sup_M H_\lambda \geq c_\lambda \geq \inf_N H_\lambda.$$

For $0 < K < 1$ and $\lambda \in [K, 1]$, let

$$\Psi_\lambda(u) = \frac{\lambda}{2} \int_{\mathbb{R}^N} V_-(x) u^2 dx - \left(\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_+(x) u^2) dx + \psi(u) \right), \quad u \in X. \quad (2.11)$$

Then $\Psi_1 = -\Phi$ and it is easy to verify that a critical point u of Ψ_λ is a weak solution of

$$-\Delta u + V_\lambda(x)u + f(x, u) = 0, \quad u \in X, \quad (2.12)$$

where

$$V_\lambda = V^+ - \lambda V^-.$$

Lemma 2.4. *Suppose that (\mathbf{v}) and $(\mathbf{f}_1) - (\mathbf{f}_3)$ hold. Then, there exist $0 < K_* < 1$ and $E \subset [K_*, 1]$ such that the Lebesgue measure of $[K_*, 1] \setminus E$ is zero and, for every $\lambda \in E$, there exist c_λ and a bounded $(PS)_{c_\lambda}$ -sequence for Ψ_λ , where c_λ satisfies*

$$+\infty > \sup_{\lambda \in E} c_\lambda \geq \inf_{\lambda \in E} c_\lambda > 0.$$

Proof. For $u \in X$, let

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} V_-(x) u^2 dx$$

and

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_+(x) u^2) dx + \psi(u).$$

Then, I and J satisfy assumptions (a) and (b) in Proposition 2.3, and, by (2.11), $\Psi_\lambda(u) = \lambda I(u) - J(u)$.

From (2.11) and (2.2), for any $u \in X$ and $\lambda \in [K, 1]$, we have

$$\begin{aligned} \Psi_\lambda(u) &= \frac{\lambda - 1}{2} \int_{\mathbb{R}^N} V_-(x) u^2 dx - \left(\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx + \int_{\mathbb{R}^N} F(x, u) dx \right) \\ &= \frac{1}{2} \|u^-\|^2 - \frac{1}{2} \|u^+\|^2 - \frac{1 - \lambda}{2} \int_{\mathbb{R}^N} V_-(x) u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx. \end{aligned} \quad (2.13)$$

Let $u_* \in X$ and $\{u_n\} \subset X$ be such that $\|u_n - u_*\| \rightarrow 0$. It follows that $u_n^- \rightarrow u_*^-$, $u_n^+ \rightarrow u_*^+$, and $u_n \rightarrow u_*$. In addition, up to a subsequence, we can assume that $u_n \rightarrow u_*$ a.e. in \mathbb{R}^N . Then, we have

$$\begin{aligned} \|u_n^-\|^2 &\rightarrow \|u_*^-\|^2, \\ \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} V_-(x) u_n^2 dx &\geq \int_{\mathbb{R}^N} V_-(x) u_*^2 dx \quad (\text{by the Fatou's lemma}), \\ \liminf_{n \rightarrow \infty} \|u_n^+\|^2 &\geq \|u_*^+\|^2. \end{aligned}$$

By Remark 1.1, $F(x, t) \geq 0$ for all x and t . This together with the Fatou's lemma implies

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_n) dx \geq \int_{\mathbb{R}^N} F(x, u_*) dx.$$

Then, by (2.13), we obtain

$$\limsup_{n \rightarrow \infty} \Psi_\lambda(u_n) \leq \Psi_\lambda(u_*).$$

This implies that Ψ_λ is τ -sequentially upper semi-continuous.

If $u_n \rightharpoonup u_*$ in X , then, for any fixed $\varphi \in X$, as $n \rightarrow \infty$,

$$\begin{aligned} \langle -\Psi'_\lambda(u_n), \varphi \rangle &= \int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + V_\lambda u_n \varphi) dx + \int_{\mathbb{R}^N} f(x, u_n) \varphi dx \\ &\rightarrow \int_{\mathbb{R}^N} (\nabla u_* \nabla \varphi + V_\lambda u_* \varphi) dx + \int_{\mathbb{R}^N} f(x, u_*) \varphi dx \\ &= \langle -\Psi'_\lambda(u_*), \varphi \rangle. \end{aligned}$$

This implies that Ψ'_λ is weakly sequentially continuous. Moreover, it is easy to see that Ψ_λ maps bounded sets to bounded sets. Therefore, Ψ_λ satisfies assumption (c) in Proposition 2.3.

Finally, we shall verify assumption (d) in Proposition 2.3 for Ψ_λ .

From assumption (f₁) and $f(x, t)/t \rightarrow 0$ as $t \rightarrow 0$ uniformly for $x \in \mathbb{R}^N$, we deduce that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$F(x, t) \leq \epsilon t^2 + C_\epsilon |t|^p, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}. \quad (2.14)$$

From (2.13) and (2.14), we have, for $u \in N$,

$$\Psi_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \frac{1-\lambda}{2} \int_{\mathbb{R}^N} V_-(x) u^2 dx - \epsilon \int_{\mathbb{R}^N} u^2 dx - C_\epsilon \int_{\mathbb{R}^N} |u|^p dx.$$

Then by the Sobolev inequality $\|u\|_{L^p(\mathbb{R}^N)} \leq C \|u\|$ and $\|u\|_{L^2} \leq C \|u\|$ (by (2.3) and (2.4)), we deduce that there exists a constant $C > 0$ such that

$$\Psi_\lambda(u) \geq \frac{1}{2} \|u\|^2 - C(1-\lambda) \max_{\mathbb{R}^N} V_-(x) \|u\|^2 - \epsilon C \|u\|^2 - C C_\epsilon \|u\|^p.$$

Choose $0 < K_* < 1$ and $\epsilon > 0$ such that $C(1-K_*) \max_{\mathbb{R}^N} V_-(x) < 1/4$ and $C\epsilon = 1/8$. Then for every $\lambda \in [K_*, 1]$, we have

$$\Psi_\lambda(u) \geq \frac{1}{8} \|u\|^2 - C C_\epsilon \|u\|^p. \quad (2.15)$$

Let $r > 0$ be such that $r^{p-2} C C_\epsilon = 1/16$ and $\beta = r^2/16$. Then from (2.15), we deduce that, for $N = \{u \in X^- \mid \|u\| = r\}$,

$$\inf_N \Psi_\lambda \geq \beta, \quad \forall \lambda \in [K_*, 1].$$

We shall prove that $\sup_{K_* \leq \lambda \leq 1} \Psi_\lambda(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$ and $u \in X^+ \oplus \mathbb{R}^+ u_0$. Arguing indirectly, assume that for some sequences $\lambda_n \in [K_*, 1]$ and $u_n \in X^+ \oplus \mathbb{R}^+ u_0$ with $\|u_n\| \rightarrow +\infty$, there is $\mathcal{L} > 0$ such that $\Psi_{\lambda_n}(u_n) \geq -\mathcal{L}$ for all n . Then, setting $w_n = u_n/\|u_n\|$, we have $\|w_n\| = 1$, and, up to a subsequence, $w_n \rightharpoonup w$, $w_n^- \rightarrow w^- \in X^-$ and $w_n^+ \rightharpoonup w^+ \in X^+$.

First, we consider the case $w \neq 0$. Dividing both sides of (2.13) by $\|u_n\|^2$, we get that

$$-\frac{\mathcal{L}}{\|u_n\|^2} \leq \frac{\Psi_{\lambda_n}(u_n)}{\|u_n\|^2} = \frac{1}{2}\|w_n^-\|^2 - \frac{1}{2}\|w_n^+\|^2 - \frac{1-\lambda_n}{2} \int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx. \quad (2.16)$$

From (1.2) and the result (a) of Lemma 2.1, we deduce that

$$\liminf_{|t| \rightarrow \infty} \frac{F(x, t)}{t^2} \geq \frac{A}{2} > \frac{1}{2} \max_{\mathbb{R}^N} V_- \geq \frac{1}{2} \mu_{-1},$$

where $A := \inf_{x \in \mathbb{R}^N, |t| \geq D} \frac{f(x, t)}{t}$. Note that for $x \in \{x \in \mathbb{R}^N \mid w \neq 0\}$, we have $|u_n(x)| \rightarrow +\infty$. This implies that, when n is large enough,

$$\int_{\{x \in \mathbb{R}^N \mid w \neq 0\}} \frac{F(x, u_n)}{u_n^2} w_n^2 dx \geq \frac{A + \mu_{-1}}{4} \int_{\{x \in \mathbb{R}^N \mid w \neq 0\}} w_n^2 dx.$$

By (1.5), we have, when n is large enough,

$$\int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx = \int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} w_n^2 dx \geq \int_{\{x \in \mathbb{R}^N \mid w \neq 0\}} \frac{F(x, u_n)}{u_n^2} w_n^2 dx.$$

Combining the above two inequalities yields

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left(\frac{1}{2} \|w_n^-\|^2 - \frac{1}{2} \|w_n^+\|^2 - \frac{1-\lambda_n}{2} \int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx \right) \\ & \leq \liminf_{n \rightarrow \infty} \left(\frac{1}{2} \|w_n^-\|^2 - \frac{1}{2} \|w_n^+\|^2 - \frac{A + \mu_{-1}}{4} \int_{\{x \in \mathbb{R}^N \mid w \neq 0\}} w_n^2 dx \right) \\ & \leq \frac{1}{2} \|w^-\|^2 - \frac{1}{2} \|w^+\|^2 - \frac{A + \mu_{-1}}{4} \int_{\mathbb{R}^N} w^2 dx \\ & \leq \frac{1}{2} \|w^-\|^2 - \frac{1}{2} \|w^+\|^2 - \frac{A + \mu_{-1}}{4} \|w^-\|_{L^2}^2. \end{aligned} \quad (2.17)$$

We used the inequalities

$$\lim_{n \rightarrow \infty} \|w_n^-\|^2 = \|w^-\|^2, \quad \liminf_{n \rightarrow \infty} \|w_n^+\|^2 \geq \|w^+\|^2 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \int_{\{x \in \mathbb{R}^N \mid w \neq 0\}} w_n^2 dx \geq \int_{\mathbb{R}^N} w^2 dx$$

in the second inequality of (2.17).

Since $w^- = t u_0$ for some $t \in \mathbb{R}$, by (2.9), we get that

$$\frac{A + \mu_{-1}}{4} \|w^-\|_{L^2}^2 \geq \frac{A + \mu_{-1}}{4\gamma} \|w^-\|^2.$$

Note that, by the choice of γ (see (2.8)), we have $\frac{A + \mu_{-1}}{4\gamma} > 1/2$. Then by (2.17) and the fact that $w \neq 0$, we have that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left(\frac{1}{2} \|w_n^-\|^2 - \frac{1}{2} \|w_n^+\|^2 - \frac{1-\lambda_n}{2} \int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx \right) \\ & \leq -\left(\frac{A + \mu_{-1}}{4\gamma} - \frac{1}{2} \right) \|w^-\|^2 - \frac{1}{2} \|w^+\|^2 < 0. \end{aligned}$$

It contradicts (2.16), since $-\mathcal{L}/\|u_n\|^2 \rightarrow 0$ as $n \rightarrow \infty$.

Second, we consider the case $w = 0$. In this case, $\lim_{n \rightarrow \infty} \|w_n^-\| = 0$. It follows that

$$\liminf_{n \rightarrow \infty} \|w_n^+\| \geq 1,$$

since $\|w_n\| = 1$ and $w_n = w_n^+ + w_n^-$. Therefore, the right hand side of (2.16) is less than $-1/4$ when n is large enough. However, as $n \rightarrow \infty$, the left hand side of (2.16) converges to zero. It induces a contradiction.

Therefore, there exists $R > r$ such that

$$\sup_{\lambda \in [K_*, 1]} \sup_{\partial M} \Psi_\lambda \leq 0.$$

This implies that Ψ_λ satisfies assumption (d) in Proposition 2.3 if $\lambda \in [K_*, 1]$. Finally, it is easy to see that

$$\sup_{\lambda \in [K_*, 1]} \sup_M \Psi_\lambda < +\infty.$$

Then, the results of this lemma follow immediately from Proposition 2.3. \square

Lemma 2.5. *Suppose that (v) and $(f_1) - (f_3)$ are satisfied. Let $\lambda \in [K_*, 1]$ be fixed, where K_* is the constant in Lemma 2.4. If $\{v_n\}$ is a bounded $(PS)_c$ sequence for Ψ_λ with $c \neq 0$, then for every $n \in \mathbb{N}$, there exists $a_n \in \mathbb{Z}^N$ such that, up to a subsequence, $u_n := v_n(\cdot + a_n)$ satisfies*

$$u_n \rightharpoonup u_\lambda \neq 0, \quad \Psi_\lambda(u_\lambda) \leq c \quad \text{and} \quad \Psi'_\lambda(u_\lambda) = 0. \quad (2.18)$$

Proof. The proof of this lemma is inspired by the proof of Lemma 3.7 in [23]. Because $\{v_n\}$ is a bounded sequence in X , up to a subsequence, either

$$(a) \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n|^2 dx = 0, \text{ or}$$

$$(b) \text{ there exist } \varrho > 0 \text{ and } a_n \in \mathbb{Z}^N \text{ such that } \int_{B_1(a_n)} |v_n|^2 dx \geq \varrho.$$

If (a) occurs, using the Lions lemma (see, for example, [25, Lemma 1.21]), a similar argument as for the proof of [23, Lemma 3.6] shows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(x, v_n) dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x, v_n) v_n^\pm dx = 0. \quad (2.19)$$

It follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (2F(x, v_n) - f(x, v_n) v_n) dx = 0. \quad (2.20)$$

On the other hand, as $\{v_n\}$ is a $(PS)_c$ sequence of Ψ_λ , we have $\langle \Psi'_\lambda(v_n), v_n \rangle \rightarrow 0$ and $\Psi_\lambda(v_n) \rightarrow c \neq 0$. It follows that

$$\begin{aligned} & \int_{\mathbb{R}^N} (f(x, v_n) v_n - 2F(x, v_n)) dx \\ &= 2\Psi_\lambda(v_n) - \langle \Psi'_\lambda(v_n), v_n \rangle \rightarrow 2c \neq 0, \quad n \rightarrow \infty. \end{aligned} \quad (2.21)$$

This contradicts (2.20). Therefore, case (a) cannot occur.

If case (b) occurs, let $u_n = v_n(\cdot + a_n)$. For every n ,

$$\int_{B_1(0)} |u_n|^2 dx \geq \varrho. \quad (2.22)$$

Because V and $F(x, t)$ are 1-periodic in every x_j , $\{u_n\}$ is still bounded in X ,

$$\lim_{n \rightarrow \infty} \Psi_\lambda(u_n) \leq c \quad \text{and} \quad \Psi'_\lambda(u_n) \rightharpoonup 0, \quad n \rightarrow \infty. \quad (2.23)$$

Up to a subsequence, we assume that $u_n \rightharpoonup u_\lambda$ in X as $n \rightarrow \infty$. Since $u_n \rightarrow u_\lambda$ in $L^2_{loc}(\mathbb{R}^N)$, it follows from (2.22) that $u_\lambda \neq 0$. Recall that $\Psi'_\lambda(u_n)$ is weakly sequentially continuous. Therefore, $\Psi'_\lambda(u_n) \rightharpoonup \Psi'_\lambda(u_\lambda)$ and, by (2.23), $\Psi'_\lambda(u_\lambda) = 0$.

Finally, by (f_3) and the Fatou's lemma

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} (\Psi_\lambda(u_n) - \frac{1}{2} \langle \Psi'_\lambda(u_n), u_n \rangle) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \tilde{F}(x, u_n) \geq \int_{\mathbb{R}^N} \tilde{F}(x, u_\lambda) = \Psi_\lambda(u_\lambda). \end{aligned}$$

□

Lemma 2.6. *There exist $0 < K_{**} < 1$ and $\eta > 0$ such that for any $\lambda \in [K_{**}, 1]$, if $u \neq 0$ satisfies $\Psi'_\lambda(u) = 0$, then $\|u\| \geq \eta$.*

Proof. We adapt the arguments of Yang [27, p. 2626] and Liu [15, Lemma 2.2]. Note that by (f_1) and (f_2) , for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(x, t)| \leq \epsilon |t| + C_\epsilon |t|^{p-1}.$$

Let $u \neq 0$ be a critical point of Ψ_λ . Then u is a solution of

$$-\Delta u + V_\lambda u + f(x, u) = 0, \quad u \in X.$$

Multiplying both sides of this equation by u^\pm respectively and then integrating on \mathbb{R}^N , we get that

$$0 = \pm \|u^\pm\|^2 + (1 - \lambda) \int_{\mathbb{R}^N} V_-(x) u_n u^\pm dx + \int_{\mathbb{R}^N} f(x, u) u^\pm dx.$$

It follows that

$$\begin{aligned} \|u^\pm\|^2 &= \mp(1 - \lambda) \int_{\mathbb{R}^N} V_-(x) u u^\pm dx \mp \int_{\mathbb{R}^N} f(x, u) u^\pm dx \\ &\leq (1 - \lambda) \sup_{\mathbb{R}^N} V_- \int_{\mathbb{R}^N} |u| \cdot |u^\pm| dx \\ &\quad + \epsilon \int_{\mathbb{R}^N} |u| \cdot |u^\pm| dx + C_\epsilon \int_{\mathbb{R}^N} |u|^{p-1} |u^\pm| dx \\ &\leq C_1((1 - \lambda) + \epsilon) \|u\| \cdot \|u^\pm\| + C_2 \|u\|^{p-1} \|u^\pm\|, \end{aligned} \tag{2.24}$$

where C_1 and C_2 are positive constants related to the Sobolev inequalities, and $\sup_{\mathbb{R}^N} V_-$. From the above two inequalities, we obtain

$$\|u\|^2 = \|u^+\|^2 + \|u^-\|^2 \leq 2C_1((1 - \lambda) + \epsilon) \|u\|^2 + 2C_2 \|u\|^p.$$

Because $p > 2$, this implies that $\|u\| \geq \eta$ for some $\eta > 0$ if $\epsilon > 0$ and $1 - K_{**} > 0$ are small enough and $\lambda \in [K_{**}, 1]$. The desired result follows. □

Let $K = \max\{K_*, K_{**}\}$, where K_* and K_{**} are the constants that appeared in Lemma 2.4 and Lemma 2.6, respectively. Combining Lemmas 2.4 – 2.6, we obtain the following lemma:

Lemma 2.7. *Suppose (v) and $(f_1) - (f_3)$ are satisfied. Then, there exist $\eta > 0$, $\{\lambda_n\} \subset [K, 1]$, and $\{u_n\} \subset X$ such that $\lambda_n \rightarrow 1$,*

$$\sup_n \Psi_{\lambda_n}(u_n) < +\infty, \quad \|u_n\| \geq \eta, \quad \text{and} \quad \Psi'_{\lambda_n}(u_n) = 0.$$

3 A priori bound of approximate solutions and proof of the main Theorem

In this section, we give a priori bound for the sequence of approximate solutions $\{u_n\}$ obtained in Lemma 2.7. We then give the proofs of Theorem 1.2.

Lemma 3.1. *Suppose (\mathbf{v}) , and $(\mathbf{f}_1) - (\mathbf{f}_3)$ are satisfied. Let $\{u_n\}$ be the sequence obtained in Lemma 2.7. Then, $\{u_n\} \subset L^\infty(\mathbb{R}^N)$ and*

$$\sup_n \|u_n\|_{L^\infty(\mathbb{R}^N)} \leq D. \quad (3.1)$$

Proof. From $\Psi'_{\lambda_n}(u_n) = 0$, we deduce that u_n is a weak solution of (2.12) with $\lambda = \lambda_n$, i.e.,

$$-\Delta u_n + V_{\lambda_n}(x)u_n + f(x, u_n) = 0 \quad \text{in } \mathbb{R}^N. \quad (3.2)$$

By assumption (\mathbf{f}_1) and the bootstrap argument of elliptic equations, we deduce that $u_n \in L^\infty(\mathbb{R}^N)$.

Multiplying both sides of (3.2) by $v_n = (u_n - D)^+$ and integrating on \mathbb{R}^N , we get that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{u_n \geq D} (V_{\lambda_n}(x)u_n + f(x, u_n))v_n dx = 0. \quad (3.3)$$

Recall that $V_{\lambda_n} = V^+ - \lambda_n V^-$ and $\lambda_n \leq 1$. Then by (1.2), we get that

$$\int_{u_n \geq D} (V_{\lambda_n}(x)u_n + f(x, u_n))v_n dx = \int_{u_n \geq D} \left(V_{\lambda_n}(x) + \frac{f(x, u_n)}{u_n} \right) u_n v_n dx \geq 0.$$

This together with (3.3) yields $v_n = 0$. It follows that $u_n(x) \leq D$ on \mathbb{R}^N .

Similarly, multiplying both sides of (3.2) by $w_n = (u_n + D)^-$ and integrating on \mathbb{R}^N , we can get that $u_n \geq -D$ on \mathbb{R}^N . Therefore, for all n , $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq D$. \square

Lemma 3.2. *Suppose that (\mathbf{v}) , (\mathbf{f}_1) , (\mathbf{f}_2) , (\mathbf{f}_3) , and (\mathbf{f}_4) are satisfied. Let $\{u_n\}$ be the sequence obtained in Lemma 2.7. Then*

$$0 < \inf_n \|u_n\| \leq \sup_n \|u_n\| < +\infty. \quad (3.4)$$

Proof. As $\Psi'_{\lambda_n}(u_n) = 0$ and $u_n \neq 0$, Lemma 2.6 implies that $\inf_n \|u_n\| > 0$.

To prove $\sup_n \|u_n\| < +\infty$, we apply an indirect argument, and assume by contradiction that $\|u_n\| \rightarrow +\infty$.

Since $\Psi'_{\lambda_n}(u_n) = 0$, by (2.24), we get that

$$\begin{aligned} \|u_n^\pm\|^2 &= \mp(1 - \lambda_n) \int_{\mathbb{R}^N} V_-(x)u_n u_n^\pm dx \mp \int_{\mathbb{R}^N} f(x, u_n)u_n^\pm dx \\ &= \mp \int_{\mathbb{R}^N} f(x, u_n)u_n^\pm dx + (1 - \lambda_n)O(\|u_n\|^2). \end{aligned}$$

It follows that

$$\begin{aligned} \|u_n\|^2 &+ \int_{\mathbb{R}^N} f(x, u_n)(u_n^+ - u_n^-)dx \\ &= \|u_n^+\|^2 + \|u_n^-\|^2 + \int_{\mathbb{R}^N} f(x, u_n)(u_n^+ - u_n^-)dx = (1 - \lambda_n)O(\|u_n\|^2). \end{aligned} \quad (3.5)$$

Set $w_n = u_n / \|u_n\|$. Then by (3.5),

$$\|u_n\|^2 \left(1 + \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx \right) = (1 - \lambda_n)O(\|u_n\|^2).$$

Then by $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$, we have that

$$\int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx \rightarrow -1, \quad n \rightarrow \infty. \quad (3.6)$$

From Lemma 2.7,

$$C_0 := \sup_n \Psi_{\lambda_n}(u_n) < +\infty.$$

Then, by $\Psi'_{\lambda_n}(u_n) = 0$, we obtain

$$2C_0 \geq 2\Psi_{\lambda_n}(u_n) - \langle \Psi'_{\lambda_n}(u_n), u_n \rangle = 2 \int_{\mathbb{R}^N} \tilde{F}(x, u_n) dx$$

From (f₃), we have

$$2C_0 \geq 2 \int_{\mathbb{R}^N} \tilde{F}(x, u_n) dx \geq 2 \int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} \tilde{F}(x, u_n) dx \quad (3.7)$$

where κ is the constant in (f₄). As the continuous function \tilde{F} is 1-periodic in every x_j variable, we deduce from (1.4) that there exists a constant $C' > 0$ such that

$$\tilde{F}(x, t) \geq C' t^2, \quad \text{for every } (x, t) \in \mathbb{R}^N \times \mathbb{R} \text{ with } \kappa \leq |t| \leq D, \quad (3.8)$$

Combining (3.7) and (3.8) leads to

$$C_0 \geq C' \int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} u_n^2 dx.$$

Dividing both sides of this inequality by $\|u_n\|^2$ and sending $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} u_n^2 dx = 0. \quad (3.9)$$

From (1.3), (2.3), and (2.4), we have that

$$\begin{aligned} & \int_{\{x \mid |u_n(x)| < \kappa\}} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx \\ & \leq \nu \int_{\{x \mid |u_n(x)| < \kappa\}} |(w_n^+ - w_n^-) w_n| dx \\ & \leq \nu \int_{\mathbb{R}^N} |(w_n^+ - w_n^-) w_n| dx \\ & \leq \nu \|w_n\|_{L^2}^2 \leq \frac{\nu}{\mu_0} \|w_n\|^2 = \frac{\nu}{\mu_0} < 1 \end{aligned} \quad (3.10)$$

where μ_0 is the constant defined in (v).

Since $f \in C(\mathbb{R}^N \times \mathbb{R})$ and $\lim_{t \rightarrow 0} f(x, t)/t = 0$, we deduce that there exists $C > 0$ such that for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \leq D$,

$$|f(x, t)| \leq C|t|.$$

This together with (3.9) gives

$$\begin{aligned} & \int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx \\ & \leq C \int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} |(w_n^+ - w_n^-) w_n| dx \\ & \leq C \|w_n^+ - w_n^-\|_{L^2} \left(\int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} w_n^2 dx \right)^{1/2} \\ & \leq C \|w_n\|_{L^2} \left(\int_{\{x \mid D \geq |u_n(x)| \geq \kappa\}} w_n^2 dx \right)^{1/2} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.11)$$

Combining (3.10) and (3.11) yields

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx \\
& \leq \limsup_{n \rightarrow \infty} \int_{\{x \mid |u_n(x)| < \kappa\}} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx \\
& \quad + \limsup_{n \rightarrow \infty} \int_{\{x \mid |u_n(x)| \geq \kappa\}} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx < 1.
\end{aligned}$$

This contradicts (3.6). Therefore, $\{u_n\}$ is bounded in X . \square

Proof of Theorem 1.2. Let $\{u_n\}$ be the sequence obtained in Lemma 2.7. From Lemma 3.2, $\{u_n\}$ is bounded in X . Therefore, up to a subsequence, either

- (a) $\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx = 0$, or
- (b) there exist $\varrho > 0$ and $y_n \in \mathbb{Z}^N$ such that $\int_{B_1(y_n)} |u_n|^2 dx \geq \varrho$.

According to (2.19), if case (a) occurs,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x, u_n) u_n^\pm dx = 0.$$

Then, by (2.24) and $\lambda_n \rightarrow 1$, we have

$$\begin{aligned}
\|u_n^\pm\|^2 &= \mp(1 - \lambda_n) \int_{\mathbb{R}^N} V_-(x) u_n u_n^\pm dx \mp \int_{\mathbb{R}^N} f(x, u_n) u_n^\pm dx \\
&\leq C(1 - \lambda_n) \|u_n\|_{L^2}^2 + \left| \int_{\mathbb{R}^N} f(x, u_n) u_n^\pm dx \right| \rightarrow 0.
\end{aligned} \tag{3.12}$$

This contradicts $\inf_n \|u_n\| > 0$ (see (3.4)). Therefore, case (a) cannot occur. As case (b) therefore occurs, $w_n = u_n(\cdot + y_n)$ satisfies $w_n \rightharpoonup u_0 \neq 0$. From (2.1) and (2.11), we have that

$$\Psi_\lambda(u) = -\Phi(u) + \frac{\lambda - 1}{2} \int_{\mathbb{R}^N} V_- u^2 dx, \quad \forall u \in X.$$

It follows that

$$\langle \Psi'_\lambda(u), \varphi \rangle = -\langle \Phi'(u), \varphi \rangle + (\lambda - 1) \int_{\mathbb{R}^N} V_- u \varphi dx, \quad \forall u, \varphi \in X. \tag{3.13}$$

By $\Psi'_{\lambda_n}(u_n) = 0$ (by Lemma 2.7), we have $\Psi'_{\lambda_n}(w_n) = 0$. From (3.13), we have that, for any $\varphi \in X$,

$$\langle \Psi'_{\lambda_n}(w_n), \varphi \rangle = -\langle \Phi'(w_n), \varphi \rangle + (\lambda_n - 1) \int_{\mathbb{R}^N} V_-(x) w_n \varphi dx.$$

Together with $\Psi'_{\lambda_n}(w_n) = 0$ and $\lambda_n \rightarrow 1$, this yields

$$\langle \Phi'(w_n), \varphi \rangle \rightarrow 0, \quad \forall \varphi \in X.$$

Finally, by $w_n \rightharpoonup u_0 \neq 0$ and the weakly sequential continuity of Φ' , we have that $\Phi'(u_0) = 0$. Therefore, u_0 is a nontrivial solution of Eq.(1.1). This completes the proof. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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